A.C.I.M FOR RANDOM INTERMITTENT MAPS : EXISTENCE, UNIQUENESS AND STOCHASTIC STABILITY

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ABSTRACT. We study a random map T which consists of intermittent maps $\{T_k\}_{k=1}^K$ and a position dependent probability distribution $\{p_{k,\varepsilon}(x)\}_{k=1}^K$. We prove existence of a unique absolutely continuous invariant measure (ACIM) for the random map T. Moreover, we show that, as ε goes to zero, the invariant density of the random system T converges in the L^1-norm to the invariant density of the deterministic intermittent map T_1. The outcome of this paper contains a first result on stochastic stability, in the strong sense, of intermittent maps.

1. INTRODUCTION

Expanding maps of the interval which admit an indifferent fixed point are good testing tools for physical systems with intermittent behaviour. In [14], Pianigiani proved the existence of ACIM for a certain class of intermittent maps of the interval. Later, polynomial decay of correlations was proved for such systems independently in [12, 15]. More recently, Hu and Vaienti generalized these results to general higher dimensional systems [11].

We are interested in perturbations of intermittent maps. In particular when the indifferent fixed point persists under perturbations. Results on statistical stability of intermittent maps with perturbations of this type were obtained in [1, 2]. More recently results on metastability [1]of intermittent maps where the neutral fixed point persists under deterministic perturbations were obtained in [7]. All the results of [1, 2, 7] are concerned with deterministic perturbations of intermittent maps.

In the random setting, i.e., when a system is randomly perturbed, if the random system admits an ACIM μ_R which converges in the weak$-*-$topology to an ACIM μ of the initial system, then we say that the system is stochastically stable in the weak sense. In addition, if the density f_R^* of μ_R converges to f^*, the density of μ, in the L^1-norm, we say the system is stochastically stable in the strong sense.

In [3] it was proved that intermittent maps of the type studied in [12] are stochastically stable in the weak sense. However, there are no results on the strong stochastic stability of such maps.

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[1]By a metastable system, we mean a system which initially has at least two ACIMs, but once it is perturbed it admits a unique ACIM. Such models were first studied in the expanding case in [10].

In this paper, we study a random map T which consists of a collection of intermittent maps $\{T_k\}_{k=1}^K$ and a probability distribution $\{p_{k,\varepsilon}(x)\}_{k=1}^K$. We prove existence of a unique ACIM for the random map T. Moreover, we show that, as ε goes to zero, the invariant density of the random system T converges in the L^1-norm to the invariant density of T_1. We obtain our results by using a cone technique. This cone was also used in [13] to study Ulam approximations for deterministic intermittent map.

In section 2, we present the setup of the problem. Section 3 contains the proof of the existence and uniqueness of the ACIM for the random map. Our main result in this section is Theorem 3.1. Section 4 contains an example of a random map which satisfy our conditions. In section 5, we show that our random maps give rise to an interesting family of 2-dimensional non-uniformly expanding maps which admit a unique ACIM. Section 6 contains the stochastic stability result. Our main result in this section is Theorem 6.1.

2. PRELIMINARIES

2.1. **Setup.** Let $(I, \mathcal{B}(I), m)$ be the measure space, where $I = [0,1], \mathcal{B}(I)$ is Borel $\sigma-$algebra and m is Lebesgue measure. To simplify the notation in the proofs, we consider a random map which consists of two maps. The proofs for any finite number of maps is similar. We study a position dependent random map

$$T = \{T_1(x), T_2(x); p_1(x), p_2(x)\}, \text{ where}$$

$$T_1 = \begin{cases} x(1+2^\alpha x^\alpha) & x \in [0, \frac{1}{2}), \\ g_1(x) & x \in [\frac{1}{2}, 1]. \end{cases} \qquad T_2 = \begin{cases} x(1+2^\beta x^\beta) & x \in [0, \frac{1}{2}), \\ g_2(x) & x \in [\frac{1}{2}, 1]. \end{cases} ,$$

where $0 < \beta < \alpha < 1$,[2,3] $g_k(\frac{1}{2}) = 0, g_k'(x) > 1, k = 1,2$ and $p_k : [0,1] \to [0,1]$ is a measurable function such that $p_1(x) + p_2(x) = 1$, i.e. $p_1(x), p_2(x)$ are position dependent probabilities. A position dependent random map is understood as a Markov process with transition function

$$\mathbb{P}(x, A) = p_1(x)\chi_A(T_1(x)) + p_2(x)\chi_A(T_2(x)),$$

where A is any measurable set in $\mathcal{B}(I)$ and χ_A is the characteristic function of the set A.

[2]Note that the assumption that $0 < \beta < \alpha < 1$ is essential for strong stochastic stability in the strong sense (convergence in L^1). If for instance $\alpha > 1$, then T_1 will admit an infinite invariant measure, i.e. this invariant measure does not have an L^1-density.

[3]The results of this paper hold for the following class of maps: Let $0 < \alpha < 1$. τ satisfying
- $\tau(0) = 0$ and there is a $t_0 \in (0,1)$ such that $\tau : [0, t_0) \to [0,1), \tau : [t_0, 1] \to [0,1]$.
- Each branch of τ is increasing, convex and is C^1; $\tau'(0) = 1$ and $\tau'(x) > 1$ for all $x \in (0, t_0) \cup (t_0, 1)$.
- There is a constant $C \in (0, \infty)$ such that $\tau(x) \geq x + Cx^{1+\alpha}$ for $x \in [0, t_0)$.

The convexity assumption is essential so that the transfer operator satisfies the cone condition $\int_0^x f dm \leq Ax^{1-\alpha} m(f)$. We choose to work with a well known representative of this family. Namely, the model studied in [12].

2.2. **Invariant measures.** The transition function $\mathbb{P}(x, A)$ induces an operator E_T on measures on $(I, \mathcal{B}(I))$ denoted by

$$\begin{aligned} E_T\mu(A) &= \int_I \mathbb{P}(x, A)d\mu(x) \\ &= \int_I p_1(x)\chi_A(T_1(x)) + p_2(x)\chi_A(T_2(x))d\mu(x) \\ &= \int_{T_1^{-1}(A)} p_1(x)d\mu(x) + \int_{T_2^{-1}(A)} p_2(x)d\mu(x). \end{aligned}$$

We say that μ is $T-$invariant if and only if

$$E_T\mu(A) = \mu(A);$$

that is, for any measurable set A,

$$\mu(A) = \int_{T_1^{-1}(A)} p_1(x)d\mu(x) + \int_{T_2^{-1}(A)} p_2(x)d\mu(x).$$

2.3. **Transfer operators.** If μ has a density function f with respect to m, then $E_T\mu$ has also a density function which we call $\mathcal{L}_T f$. We obtain, for any measurable set A,

$$\begin{aligned} \int_A \mathcal{L}_T f dm(x) &= E_T\mu(A) = \int_{T_1^{-1}(A)} p_1(x)d\mu(x) + \int_{T_2^{-1}(A)} p_2(x)d\mu(x) \\ &= \int_{T_1^{-1}(A)} p_1(x)f dm(x) + \int_{T_2^{-1}(A)} p_2(x)f dm(x) \\ &= \int_A p_1(T_1^{-1}(x))f(T_1^{-1}(x))dm(x) + \int_A p_2(T_2^{-1}(x))f(T_2^{-1}(x))dm(x) \\ &= \int_A P_{T_1}(p_1 f)dm(x) + \int_A P_{T_2}(p_2 f)dm(x) \\ &= \int_A [P_{T_1}(p_1 f) + P_{T_2}(p_2 f)]dm(x), \end{aligned} \tag{2.1}$$

where P_{T_1} and P_{T_2} are Perron-Frobenius operators [8] associated with T_1 and T_2 respectively. Since (2.1) holds for any measurable set A, we will get an almost everywhere equality: [4]

$$\begin{aligned} (\mathcal{L}_T f)(x) &= P_{T_1}(p_1 f)(x) + P_{T_2}(p_2 f)(x) \\ &= \sum_{y\in T_1^{-1}(x)} \frac{(p_1 f)(y)}{|T_1'(y)|} + \sum_{y\in T_2^{-1}(x)} \frac{(p_2 f)(y)}{|T_2'(y)|}. \end{aligned}$$

We call $\mathcal{L}_T$ the Perron-Frobenius operator associated with the random map T. The properties of $\mathcal{L}_T$ resemble the properties of the classical Perron-Frobenius

[4]Note that since $p_1(x), p_2(x)$ are functions of x, $\mathcal{L}_T$ is not a convex combination of p_1 and p_2.

operator associated with a single deterministic map. $\mathcal{L}_T$ satisfies the properties as follows (See [5] Lemma 3.1):
(i)(Linearity) $\mathcal{L}_T : L^1 \to L^1$ is a linear operator.
(ii)(Positivity) Let $f \in L^1$ and assume $f \geq 0$, then $\mathcal{L}_T f \geq 0$.
(iii)(Preservation of integrals)

$$\int_I \mathcal{L}_T f dm(x) = \int_I f dm(x)$$

(iv)(contraction) for any $f \in L^1$,

$$\| \mathcal{L}_T f \|_1 \leq \| f \|_1$$

(v) $\mathcal{L}_T f = f \Leftrightarrow E_T \mu = \mu$, i.e measure $\mu = f \cdot m$ is $T-$invariant.
(vi)(composition)

$$\mathcal{L}_{T \circ R} f = \mathcal{L}_T \circ \mathcal{L}_R f$$

In particular, $\mathcal{L}_{T^n} f = \mathcal{L}_T^n f$.

2.4. **Notation.** For $x \in I, k \in \{1,2\}$ and partition $\mathcal{P} = \{I_1, I_2\}, I_1 = [0, \frac{1}{2}], I_2 = [\frac{1}{2}, 1]$, we introduce the following definitions

$$\begin{aligned} T(x) &= T_k(x), \quad \text{with probability } p_k(x) \\ T^n(x) &= T_{k_n} \circ T_{k_{n-1}} \circ \cdots \circ T_{k_1}(x), \text{ with probability} \\ & \quad p_{k_n}(T_{k_{n-1}} \circ \cdots \circ T_{k_1}(x)) \cdot p_{k_{n-1}}(T_{k_{n-2}} \circ \cdots \circ T_{k_1}(x)) \cdot p_{k_1}(x), \quad k_i \in \{1,2\} \\ T_{k,i} &= T_k \mid_{I_i} . \end{aligned}$$

We write

$$T_1^{-1} x = \{y_1, z_1\}, \quad y_1 \leq \frac{1}{2} \leq z_1;$$
$$T_2^{-1} x = \{y_2, z_2\}, \quad y_2 \leq \frac{1}{2} \leq z_2;$$
$$y_* = \max\{y_1, y_2\} \in [0, \frac{1}{2}], \quad z_* = \max\{z_1, z_2\} \in [\frac{1}{2}, 1];$$

and $m(f) = \int_0^1 f(x) dm(x)$, where m is Lebesgue measure.

Cone. For $A > 0$, define

$$\mathcal{C}_A = \{ f \in L^1 \mid f \geq 0, f \text{ decreasing}, \int_0^x f dm \leq A x^{1-\alpha} m(f) \}.$$

3. EXISTENCE AND UNIQUENESS OF ACIM

3.1. **Sufficient conditions for the existence of a $T-$ACIM.** For $k = 1, 2$, we assume

(A) $\sum_{i=1}^{l} \frac{p_k(T_{k,i}^{-1}(x))}{T_k'(T_{k,i}^{-1}(x))}, 1 \leq l \leq 2$, is decreasing;
(B) $\inf_{x \in I} p_k(x) \geq \delta > 0$.

Theorem 3.1. *Under assumptions (A) and (B)*
(i) The random map T admits a unique ACIM $\mu, d\mu = \rho dm$.
(ii) The invariant density ρ is uniformly bounded below.

We first prove some technical lemmas. The proof of the Theorem 3.1 is at the end of this section.

Lemma 3.2. *Let $f \in \mathcal{C}_A$. Then, for $x \in (0,1]$,*
(i) $f(x) \le Ax^{-\alpha} m(f)$;
(ii) $f(x) \le \frac{1}{x} m(f)$*, and in particular,* $f(x)|_{x\in[\frac{1}{2},z_*)} \le 2m(f)$;
(iii) $y_1 \ge \frac{x}{2}, y_2 \ge \frac{x}{2}$ *and* $x \ge y_*$;
(iv) $(1-x)^{1-\alpha} \le 1-(1-\alpha)x$;
(v) $x^{1-\alpha} - y_*^{1-\alpha} \ge \frac{1-\alpha}{2} x$.

Proof. (i) We have

$$xf(x) = \int_0^x f(x) dm(\xi) \le \int_0^x f(\xi) dm(\xi) \le Ax^{1-\alpha} m(f).$$

(ii) By $f(x) \ge 0$ and decreasing, we have

$$xf(x) = \int_0^x f(x) dm(\xi) \le \int_0^x f(\xi) dm(\xi) \le \int_0^x f(\xi) dm(\xi) + \int_x^1 f(\xi) dm(\xi) = m(f).$$

So, $f(x) \le \frac{1}{x} m(f)$ and in particular $f(x) \le 2m(f)$, when $x \in [\frac{1}{2}, z_*)$.

(iii) For $y_1, y_2 \le \frac{1}{2}, 0 < \beta < \alpha < 1$, we have

$$x = T_1(y_1) = y_1(1+2^\alpha y_1^\alpha) \le 2y_1 \text{ and } x = T_2(y_2) = y_2(1+2^\alpha y_2^\beta) \le 2y_2.$$

Also,

$$x = T_1(y_1) = y_1(1+2^\alpha y_1^\alpha) \ge y_1 \text{ and } x = T_2(y_2) = y_2(1+2^\alpha y_2^\beta) \ge y_2.$$

Therefore, $y_1 \ge \frac{x}{2}, y_2 \ge \frac{x}{2}$ and $x \ge y_*$.

(iv) Set

$$g(x) = (1-x)^{1-\alpha} - [1-(1-\alpha)x],$$

then $g(0) = 1-1 = 0$ and for $x \in [0,1]$,

$$g'(x) = -(1-\alpha)(1-x)^{-\alpha} + (1-\alpha) = (1-\alpha)[1 - \frac{1}{(1-x)^\alpha}] \le 0.$$

Therefore, $g(x) \le 0, x \in (0,1]$, that is $(1-x)^{1-\alpha} \le 1-(1-\alpha)x$.

(v) First write,

$$x^{1-\alpha} - y_*^{1-\alpha} = x^{1-\alpha}[1 - (\frac{y_*}{x})^{1-\alpha}] = x^{1-\alpha}[1 - (1 - \frac{x-y_*}{x})^{1-\alpha}].$$

Let $\zeta = \frac{x-y_*}{x}$.

In case $y_* = y_1$,

$$x = T_1(y_1) = y_1(1+2^\alpha y_1^\alpha) > y_1 > 0, \quad x \le 2y_1 \quad \text{and } \zeta = \frac{x-y_1}{x} \in (0,1].$$

Thus,

$$\begin{aligned}
x^{1-\alpha} - y_*^{1-\alpha} &= x^{1-\alpha}[1-(1-\zeta)^{1-\alpha}] \\
&\geq x^{1-\alpha}[1-(1-(1-\alpha)\zeta)] \\
&= x^{1-\alpha}(1-\alpha)\frac{x-y_1}{x} \\
&= x^{-\alpha}(1-\alpha)(T_1(y_1)-y_1) \\
&= x^{-\alpha}(1-\alpha)(2^\alpha y_1^{\alpha+1}) \\
&\geq (2y_1)^{-\alpha}(1-\alpha)(2^\alpha y_1^{\alpha+1}) \\
&= (1-\alpha)y_1 \\
&\geq \frac{(1-\alpha)}{2}x.
\end{aligned}$$

In case $y_* = y_2$,

$$x = T_2(y_2) = y_2(1+2^\beta y_2^\beta) > y_2 > 0, \quad x \leq 2y_2 \quad \text{and } \zeta = \frac{x-y_2}{x} \in (0,1].$$

We have

$$\begin{aligned}
x^{1-\alpha} - y_*^{1-\alpha} &= x^{1-\alpha}[1-(1-\zeta)^{1-\alpha}] \\
&\geq x^{1-\alpha}[1-(1-(1-\alpha)\zeta)] \\
&= x^{1-\alpha}(1-\alpha)\frac{x-y_2}{x} \\
&= x^{-\alpha}(1-\alpha)(T_2(y_2)-y_1) \\
&= x^{-\alpha}(1-\alpha)(2^\beta y_2^{\beta+1}) \\
&\geq (2y_2)^{-\alpha}(1-\alpha)(2^\beta y_2^{\beta+1}) \\
&= (1-\alpha)y_2(2y_2)^{\beta-\alpha}.
\end{aligned}$$

Since $0 < \beta < \alpha < 1$ and $0 \leq 2y_2 \leq 1$, we get $(2y_2)^{\beta-\alpha} \geq 1$.
Thus,

$$x^{1-\alpha} - y_*^{1-\alpha} \geq (1-\alpha)y_2 \geq (1-\alpha)\frac{x}{2}.$$

□

Lemma 3.3. *Let $f \geq 0$ be a decreasing function. Then $\mathcal{L}_T f$ is also decreasing.*

Proof. See Lemma 3.1 of [6].[5] □

Proposition 3.4. *For $A \geq \frac{4}{1-\alpha}$ the cone $\mathcal{C}_A$ is invariant under the action of the operator $\mathcal{L}_T$.*

Proof. By Lemma 3.3, for $f \in \mathcal{C}_A$ we know that $\mathcal{L}_T f$ is decreasing . Also, $\mathcal{L}_T f \geq 0$ and $m(\mathcal{L}_T f) = m(f)$. Therefore we only need to prove that

$$\int_0^x \mathcal{L}_T f dm \leq A x^{1-\alpha} m(\mathcal{L}_T) = A x^{1-\alpha} m(f),$$

[5] Note that this Lemma only requires assumption (A) to hold.

when $A \geq A_* = \frac{4}{1-\alpha}$. We have

$$\begin{aligned}
\int_0^x \mathcal{L}_T f dm &= \int_0^x P_{T_1}(p_1 f) + P_{T_2}(p_2 f) dm = \int_{T_1^{-1}[0,x]} (p_1 f) dm + \int_{T_2^{-1}[0,x]} (p_2 f) dm \\
&= (\int_0^{y_1} + \int_{\frac{1}{2}}^{z_1})(p_1 f) dm + (\int_0^{y_2} + \int_{\frac{1}{2}}^{z_2})(p_2 f) dm \\
&\leq (\int_0^{y_*} + \int_{\frac{1}{2}}^{z_*})(p_1 f) dm + (\int_0^{y_*} + \int_{\frac{1}{2}}^{z_*})(p_2 f) dm \\
&= \int_0^{y_*} (p_1 + p_2) f dm + \int_{\frac{1}{2}}^{z_*} (p_1 + p_2) f dm \leq A y_*^{1-\alpha} m(f) + \int_{\frac{1}{2}}^{z_*} f dm,
\end{aligned}$$

where $y_* = \max\{y_1, y_2\} \in [0, \frac{1}{2}]$, $\quad z_* = \max\{z_1, z_2\} \in [\frac{1}{2}, 1]$. Since our transformations $T_k(x)$ may not be piecewise onto, there are two cases to consider.

In the case 1, x has only one pre-image. By the Lemma 3.2 (iii), we get $x \geq y_*$. So, $y_*^{1-\alpha} \leq x^{1-\alpha}$, with $1 - \alpha > 0$. Therefore, $\int_0^x \mathcal{L}_T f dm \leq A y_*^{1-\alpha}$ for $A > 0$.

In the case 2, x has two preimages. From Lemma 3.2, we have $f(x) \leq 2m(f), x \in [\frac{1}{2}, z_*]$. Then,

$$\int_{\frac{1}{2}}^{z_*} f dm \leq \int_{\frac{1}{2}}^{z_*} 2m(f) dm = 2(z_* - \frac{1}{2}) m(f).$$

Moreover, we have $g_1'(x) > 1, g_2'(x) > 1$, then $x = m[0, x] = m \circ T_k[\frac{1}{2}, z_k] \geq z_k - \frac{1}{2}, k = 1, 2$ i.e. $x > z_* - \frac{1}{2}$. So,

$$\int_{\frac{1}{2}}^{z_*} f dm < 2x m(f).$$

By the result of Lemma 3.2 (iv), we obtain that $x \leq \frac{2}{1-\alpha}(x^{1-\alpha} - y_*^{1-\alpha})$. Then, for $A \geq A_* = \frac{4}{1-\alpha}$,

$$\int_0^x \mathcal{L}_T f dm < A y_*^{1-\alpha} m(f) + \frac{4}{1-\alpha}(x^{1-\alpha} - y_*^{1-\alpha}) m(f) \leq A x^{1-\alpha} m(f).$$

Therefore, $\mathcal{L}_T f \in \mathcal{C}_A$, for $f \in \mathcal{C}_A$ and $A \geq \frac{4}{1-\alpha}$. $\square$

Remark 3.5. Obviously, if $f \in \mathcal{C}_A$ and $A \geq \frac{4}{1-\alpha}$, then $\mathcal{L}_T^n f \in \mathcal{C}_A, n \geq 1$.

Remark 3.6. Since $\mathcal{C}_A$ is compact and convex, operator $\mathcal{L}_T$ has a fixed point $f_* \in \mathcal{C}_A$ by Proposition 3.4 and the Schauder-Tychonoff fixed point theorem of [9]. Thus, random map T admits an ACIM .

Let μ be an ACIM for random map T. Each of the maps T_k admits a unique ACIM (See Appendix). Let ν_1 and ν_2 be the unique ACIM for T_1 and T_2 respectively. Let $A_k = \text{supp}(\nu_k)$ and $\mathcal{U}_k = \bigcup_{j=0}^{\infty} T_k^{-j} A_k$ be its basin. For $k = 1, 2$, we have $A_k = \mathcal{U}_k = I$ (see Appendix).

Lemma 3.7. *For $k = 1, 2, I = A_k \subseteq supp(\mu)$.*

Proof. Since $A_k = \mathcal{U}_k = I$ for $k = 1, 2$. Then $\mu(A_k) = \mu(\mathcal{U}_k) > 0$.

Let $B = I \cap \text{supp}(\mu)$, then $B \neq \emptyset$ and $\mu(B) > 0$. Since B is subset of $I = A_k$ and A_k is an invariant set, then $\bigcup_{i=0}^{\infty} T_k^i B \subseteq A_k$.

Assume $A_k \nsubseteq \text{supp}(\mu)$. Then $\mu(A_k \setminus B) = 0$. Also,

$$\mu(A_k \setminus B) \geq \mu(\bigcup_{i=0}^{\infty} T_k^i B \setminus B) = \mu(\bigcup_{i=1}^{\infty} T_k^i B \setminus B) \geq \mu(T_k^i B), \quad i = 1, 2, ...$$

So, in this case, $\mu(T_k^i B) \leq 0, i = 1, 2, ...$ However this leads to a contradiction because, by condition (B),

$$\begin{aligned} \mu(T_1 B) &= \int_{T_1^{-1}(T_1 B)} p_1 d\mu + \int_{T_2^{-1}(T_1 B)} p_2 d\mu \\ &\geq \inf_{x \in I} p_1(x)\mu(B) + \inf_{x \in I} p_2(x)\mu(T_2^{-1}(T_1 B)) > 0. \end{aligned}$$

and

$$\begin{aligned} \mu(T_2 B) &= \int_{T_1^{-1}(T_2 B)} p_1 d\mu + \int_{T_2^{-1}(T_2 B)} p_2 d\mu \\ &\geq \inf_{x \in I} p_1(x)\mu(T_1^{-1}(T_2 B)) + \inf_{x \in I} p_2(x)\mu(B) > 0 \end{aligned}$$

Therefore, $I = A_k \subseteq \text{supp}(\mu)$. □

Proposition 3.8. *Let $A \geq A_* = \frac{4}{1-\alpha}$ and $f \in \mathcal{C}_A$. There are $\gamma > 0, N \in \mathbb{Z}_+$ such that $\mathcal{L}_T^n f \geq \gamma m(f)$, for all $n \geq N$, where γ and N depend only on A. In particular, if $\rho = \mathcal{L}_T \rho$ then $\mu = \rho m$ is equivalent to m.*

Proof. First by Proposition 3.4, if $A \geq \frac{4}{1-\alpha}, f \in \mathcal{C}_A$, then $\mathcal{L}_T^n f \in \mathcal{C}_A$. So, we have

$$\int_0^x f dm \leq A x^{1-\alpha} m(f), \int_0^x \mathcal{L}_T^n f dm \leq A x^{1-\alpha} m(\mathcal{L}_T^n f).$$

Without loss of generality, we suppose that $m(f) = 1$. Then $m(\mathcal{L}_T^n f) = m(f) = 1$. Therefore we only need to prove $\mathcal{L}_T^n f \geq \gamma$. Fix a small number $0 < \sigma < \frac{1}{2}$, such that $A\sigma^{1-\alpha} = \frac{1}{2}$. Then,

$$\int_0^\sigma f dm \leq A\sigma^{1-\alpha} = \frac{1}{2} \text{ and } \int_\sigma^1 f dm = 1 - \int_0^\sigma f dm \geq \frac{1}{2}.$$

When $x \in (0, \sigma)$, since $f(x)$ is a decreasing function, we have

$$f(x) \geq f(\sigma) = \frac{\int_\sigma^1 f(\sigma)dm}{1-\sigma} \geq \frac{\int_\sigma^1 f(x)dm}{1-\sigma} \geq \frac{1}{2(1-\sigma)}.$$

Moreover, $\mathcal{L}_T^n f(x)$ is decreasing. Then it is enough to show that $\mathcal{L}_T^n f(1)$ is bounded below away from zero. By (vi) composition property of $\mathcal{L}_T$ we have $\mathcal{L}_T^n f(1) = \mathcal{L}_{T^n} f(1)$. We will show that $\mathcal{L}_{T^n} f(1) \geq \gamma > 0$. Set $\omega_n = \{k_1, k_2, ..., k_n \in \{1,2\}^n\}, k_i \in \{1,2\}$. Define $x_n = T_k^{-1}(x_{n-1}) \cap [0, \frac{1}{2}], n \geq 1$ and $x_0 = 1$. Obviously, $\{x_n\}$ is a strictly decreasing sequence and it converges to 0. Since $\{x_n\}$ depends on ω_n, we denote $\{x_{n,\omega_n}\} = \{x_n\}(\omega_n)$. With the fixed σ, we can find an N such that $\{0, b_1, b_2, ..., b_q\}$ are critical points of map T_{ω_N} and

$$\{b_1, b_2\} = T_{k_1}^{-1}(x_{N-1}), \quad \max_{\omega_N} x_{N-1,\omega_N} \leq \sigma.$$

Then, for all ω_N, we have $\mathcal{L}_T f(x_{N-1,\omega_N}) \geq \mathcal{L}_T f(\sigma)$ since $\mathcal{L}_T f(x)$ is decreasing.

(3.1)

$$\begin{aligned}
\mathcal{L}_{T^N} f(1) = & \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{q} \frac{(p_{\omega_N} f)(b_i)}{T'_{\omega_N}(b_i)} \\
= & \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{q} \frac{p_{\omega_{N-1}}(T_{k_1}(b_i)) p_{k_1}(b_i) f(b_i)}{T'_{\omega_{N-1}}(T_{k_1}(b_i)) T'_{k_1}(b_i)} \\
\geq & \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{2} \frac{p_{\omega_{N-1}}(T_{k_1}(b_i)) p_{k_1}(b_i) f(b_i)}{T'_{\omega_{N-1}}(T_{k_1}(b_i)) T'_{k_1}(b_i)} \\
= & \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{2} \frac{p_{\omega_{N-1}}(x_{N-1}) p_{k_1}(T_{k_1,i}^{-1} x_{N-1}) f(T_{k_1,i}^{-1} x_{N-1})}{T'_{\omega_{N-1}}(x_{N-1}) T'_{k_1}(T_{k_1,i}^{-1} x_{N-1})} \\
= & \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \sum_{k_1=1}^{2} \frac{p_{\omega_{N-1}}(x_{N-1})}{T'_{\omega_{N-1}}(x_{N-1})} \left(\sum_{i=1}^{2} \frac{p_{k_1}(T_{k_1,i}^{-1} x_{N-1}) f(T_{k_1,i}^{-1} x_{N-1})}{T'_{k_1}(T_{k_1,i}^{-1} x_{N-1})} \right) \\
= & \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})} \left[\sum_{k_1=1}^{2} \sum_{i=1}^{2} \frac{p_{k_1}(T_{k_1,i}^{-1} x_{N-1,\omega_N}) f(T_{k_1,i}^{-1} x_{N-1,\omega_N})}{T'_{k_1}(T_{k_1,i}^{-1} x_{N-1,\omega_N})} \right] \\
= & \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})} [\mathcal{L}_T f(x_{N-1,\omega_N})] \\
\geq & \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})} [\mathcal{L}_T f(\sigma)].
\end{aligned}$$

We have $\max\limits_{k \in \{1,2\}, x \in [0, \frac{1}{2}]} T'_k(x) = 2 + \alpha$ and $f(T_{k,1}^{-1}\sigma) > f(\sigma) \geq \frac{1}{2(1-\sigma)}$, and by condition (B): $\inf p_k(x) \geq \delta > 0$. Therefore, from (3.1) it remains to show that $\mathcal{L}_T f(\sigma) > 0$. Indeed,

$$\begin{aligned}\mathcal{L}_T f(\sigma) &= \frac{p_1(T_{k,1}^{-1}\sigma)f(T_{k,1}^{-1}\sigma)}{T_k'(T_{k,1}^{-1}\sigma)} + \frac{p_2(T_{k,2}^{-1}\sigma)f(T_{k,2}^{-1}\sigma)}{T_k'(T_{k,2}^{-1}\sigma)} \\ &\geq \frac{p_1(T_{k,1}^{-1}\sigma)f(T_{k,1}^{-1}\sigma)}{T_k'(T_{k,1}^{-1}\sigma)} \geq \frac{\delta}{2(1-\sigma)(2+\alpha)} > 0.\end{aligned}$$

Therefore,

$$\mathcal{L}_T^N f(x) \quad \geq \quad \mathcal{L}_{T^N} f(1) \geq \gamma > 0,$$

where $\gamma = \frac{\delta}{2(1-\sigma)(2+\alpha)} \sum\limits_{\omega_{N-1}\in\{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})}$ with N, σ depending only on A. Moreover, for $n > N$, we set $h(x) = \mathcal{L}_T^{n-N} f(x)$. Then $h(x) \in \mathcal{C}_A$, and

$$\mathcal{L}_T^n f(x) \quad = \quad \mathcal{L}_T^N(\mathcal{L}_T^{n-N} f(x)) = \mathcal{L}_T^N h(x) \geq \gamma.$$

Thus, for all $n \geq N, \mathcal{L}_T^n f(x) \geq \gamma > 0$. For last part of the proposition, suppose that $\rho = \mathcal{L}_T\rho \in \mathcal{C}_A$. Clearly, if set E such that $m(E) = 0$, it follows that $\mu(E) = \int_E \rho dm = 0$. Conversely, $\mu(E) = 0$. $\rho = \mathcal{L}_T^n\rho$ implies that $0 = \mu(E) = \int_E \rho dm = \int_E \mathcal{L}_T^n\rho dm \geq \gamma m(E)$. Hence, if $\rho = \mathcal{L}_T\rho$ then $\mu = \rho m$ is equivalent to m. □

Proof. (Theorem 3.1) Since $\mathcal{C}_A$ is compact and convex, operator $\mathcal{L}_T$ has a fixed point $f_* \in \mathcal{C}_A$ by Proposition 3.4 and the Schauder-Tychonoff fixed point theorem of [9]. Thus, random map T admits an ACIM. Next, we give the proof of uniqueness. Suppose that the random map T has two mutually singular ACIM μ_1 and μ_2. From Lemma 3.6, we have $I = A_k \subseteq \text{supp}(\mu_1)$ and $I = A_k \subseteq \text{supp}(\mu_2)$. Therefore, $I \subseteq \text{supp}(\mu_1) \cap \text{supp}(\mu_2)$. This contradicts the mutual singularity of μ_1 and μ_2. Thus, the random map T has a unique ACIM. By Proposition 3.7, the invariant density ρ is uniformly bounded below. □

4. EXAMPLE

We present an example of a random map T which satisfies assumptions (A) and (B). Consequently, by Theorem 3.1 this random map has a unique ACIM.

Example 4.1. Let random map $T = \{T_1(x), T_2(x); p_1(x), p_2(x)\}$, for $0 < \beta < \alpha < 1$,

$$T_1 = \begin{cases} x(1+2^\alpha x^\alpha) & x \in [0, \frac{1}{2}), \\ 2x - 1 & x \in [\frac{1}{2}, 1]. \end{cases} \qquad T_2 = \begin{cases} x(1+2^\beta x^\beta) & x \in [0, \frac{1}{2}), \\ \frac{3}{2}x - \frac{3}{4} & x \in [\frac{1}{2}, 1]. \end{cases}$$

and

$$p_1 = \begin{cases} \frac{1+x^\alpha}{3} & x \in [0, \frac{1}{2}), \\ \frac{1}{3} & x \in [\frac{1}{2}, 1]. \end{cases} \qquad p_2 = \begin{cases} \frac{2-x^\alpha}{3} & x \in [0, \frac{1}{2}), \\ \frac{2}{3} & x \in [\frac{1}{2}, 1]. \end{cases} .$$

We have $p_1(x), p_2(x) \in [0,1], p_1(x) + p_2(x) = 1, \forall x \in [0,1]$ and $\inf\limits_{x\in I} p_k(x) \geq \frac{1}{3} > 0$. Thus, condition (B) is satisfied. We now check condition (A). First, for $x \in [0, \frac{1}{2})$, $\frac{p_2(x)}{T_2'(x)}$ is obviously decreasing. For $\frac{p_1(x)}{T_1'(x)}$, we show

$$p_1^{'}(x)T_1^{'}(x) - p_1(x)T_1^{''}(x) \leq 0, \forall x \in [0, \frac{1}{2}).$$

$$\begin{aligned} p_1'(x)T_1'(x) - p_1(x)T_1''(x) &= \frac{1}{3}\alpha x^{\alpha-1}[1+(1+\alpha)2^\alpha x^\alpha] - \frac{1+x^\alpha}{3}[\alpha 2^\alpha (1+\alpha)x^{\alpha-1}] \\ &= \frac{\alpha x^{\alpha-1}}{3}[1+(1+\alpha)2^\alpha x^\alpha - (1+x^\alpha)2^\alpha(1+\alpha)] \\ &= \frac{\alpha x^{\alpha-1}}{3}[1-2^\alpha(1+\alpha)]. \end{aligned}$$

The term in square bracket is negative, i.e. $1 < 2^\alpha(1+\alpha), \forall \alpha \in (0,1)$.
So, $\frac{p_1(x)}{T_1'(x)}$ is decreasing since

$$(\frac{p_1(x)}{T_1'(x)})' = \frac{p_1'(x)T_1'(x) - p_1(x)T_1''(x)}{(T_1'(x))^2} \leq 0.$$

For $x \in [\frac{1}{2}, 1]$, $\frac{p_1(x)}{T_1'(x)} = \frac{1}{6}$, $\frac{p_2(x)}{T_2'(x)} = \frac{4}{9}$. Therefore, $\sum_{i=1}^{l} \frac{p_k(T_{k,i}^{-1}(x))}{T_k'(T_{k,i}^{-1}(x))}, 1 \leq l \leq 2$, is decreasing for all $k = 1, 2$, since $x \mapsto T_{1,1}^{-1}x$ and $x \mapsto T_{2,1}^{-1}x$ are increasing. This random map preserves a unique ACIM.

5. TWO DIMENSIONAL NON-UNIFORMLY EXPANDING MAP

In this section we use the skew product representation of [4] and show that our random maps give rise to an interesting family of 2-dimensional non-uniformly expanding maps which admit a unique ACIM. This family could serve as a good testing tool for the analysis of 2-dimensional systems with slow mixing.

Let $S(x,\omega) : I^2 \to I^2$ be

$$S(x,\omega) = (T_k(x), \varphi_{k,x}(\omega)), \text{ where } \begin{cases} \varphi_{1,x}(\omega) = \frac{\omega}{p_1(x)}, & \omega \in [0, p_1(x)), \\ \varphi_{2,x}(\omega) = \frac{\omega - p_1(x)}{p_2(x)}, & \omega \in [p_1(x), 1]. \end{cases}$$

Define $S_i = S\,|_{U_i}, i = 1,2,3,4$.

$S_1 = (T_{1,1}(x), \varphi_{1,x}(\omega)) = (x(1+2^\alpha x^\alpha), \frac{\omega}{p_1(x)})$, $\quad U_1 = [0, \frac{1}{2}) \times [0, p_1(x))$,

$S_2 = (T_{1,2}(x), \varphi_{1,x}(\omega)) = (g_1(x), \frac{\omega}{p_1(x)})$, $\quad U_2 = [\frac{1}{2}, 1] \times [0, p_1(x))$,

$S_3 = (T_{2,2}(x), \varphi_{2,x}(\omega)) = (g_2(x), \frac{\omega - p_1(x)}{p_2(x)})$, $\quad U_3 = [\frac{1}{2}, 1] \times [p_1(x), 1]$,

$S_4 = (T_{2,1}(x), \varphi_{2,x}(\omega)) = (x(1+2^\beta x^\beta), \frac{\omega - p_1(x)}{p_2(x)})$, $\quad U_4 = [0, \frac{1}{2}) \times [p_1(x), 1]$.

One can easily check that $(0,0)$ is a fixed point of S. Moreover, the lyapunov exponent in the horizontal direction has value zero at $(0,0)$. Therefore, S is nonuniformly expanding map. Moreover, under conditions (A) and (B), since T has unique ACIM, by [4], S has a unique ACIM too.

6. STOCHASTIC STABILITY

In this section we study stochastic stability of random intermittent maps. For this purpose, we write, for $\varepsilon > 0, 0 < \alpha - \varepsilon < 1$,

$$T_\varepsilon = \{T_1(x), T_{1,\varepsilon}(x); p_{1,\varepsilon}(x), p_{2,\varepsilon}(x)\},$$

where

$$T_1 = \begin{cases} x(1+2^\alpha x^\alpha) & x \in [0, \frac{1}{2}), \\ g_1(x) & x \in [\frac{1}{2}, 1]. \end{cases} \qquad T_{1,\varepsilon} = \begin{cases} x(1+2^{\alpha-\varepsilon} x^{\alpha-\varepsilon}) & x \in [0, \frac{1}{2}), \\ g_{1,\varepsilon}(x) & x \in [\frac{1}{2}, 1]. \end{cases} .$$

Our main result in this section is the following theorem.

Theorem 6.1. *Let f_ε be the unique invariant density of T_ε. Let f^* be the unique invariant density of T_1. If $\lim_{\varepsilon\to 0}\sup_x p_{2,\varepsilon}(x)=0$, then $\lim_{\varepsilon\to 0}\|f_\varepsilon - f^*\|_1 = 0$.*

Proof. Let $\mathcal{L}_{T_\varepsilon}$ be the Perron-Frobenius operator associated with the random map T_ε. By Theorem 3.1, there exist a fixed point f_ε of $\mathcal{L}_{T_\varepsilon}$ and $f_\varepsilon \in \mathcal{C}_A$, for some $A \geq \frac{4}{1-\alpha}$. Since $\mathcal{C}_A$ is a compact set, there exists a subsequence $\{f_{\varepsilon_k}\}_{\varepsilon_k>0}$ of $\{f_\varepsilon\}_{\varepsilon>0}$ such that

$$f_{\varepsilon_k} \xrightarrow{L^1} f^* \in \mathcal{C}_A, \text{ as } \varepsilon_k \to 0.$$

We have

$$\begin{aligned}
\|f^* - P_{T_1}f^*\|_1 &\leq \|f^* - f_{\varepsilon_k}\|_1 + \|f_{\varepsilon_k} - P_{T_1}f_{\varepsilon_k}\|_1 + \|P_{T_1}f_{\varepsilon_k} - P_{T_1}f^*\|_1\\
&\leq \|f^* - f_{\varepsilon_k}\|_1 + \|f_{\varepsilon_k} - P_{T_1}f_{\varepsilon_k}\|_1 + \|f_{\varepsilon_k} - f^*\|_1\\
&= 2\|f^* - f_{\varepsilon_k}\|_1 + \|\mathcal{L}_{T_{\varepsilon_k}}f_{\varepsilon_k} - P_{T_1}f_{\varepsilon_k}\|_1.
\end{aligned}$$

The first term on the right converges to 0 as $\varepsilon_k \to 0$ by the choice of subsequence. Moreover, we have

$$\begin{aligned}
\|\mathcal{L}_{T_{\varepsilon_k}}f_{\varepsilon_k} - P_{T_1}f_{\varepsilon_k}\|_1 &= \|P_{T_1}(p_{1,\varepsilon_k}f_{\varepsilon_k}) + P_{T_{1,\varepsilon}}(p_{2,\varepsilon_k}f_{\varepsilon_k}) - P_{T_1}f_{\varepsilon_k}\|_1\\
&= \|P_{T_1}(p_{1,\varepsilon_k}f_{\varepsilon_k} - f_{\varepsilon_k}) + P_{T_{1,\varepsilon}}(p_{2,\varepsilon_k}f_{\varepsilon_k})\|_1\\
&= \|(P_{T_{1,\varepsilon}} - P_{T_1})p_{2,\varepsilon_k}f_{\varepsilon_k}\|_1\\
&\leq 2\|p_{2,\varepsilon_k}f_{\varepsilon_k}\|_1 \leq 2\sup_x p_{2,\varepsilon_k} \to 0, \text{ as } \varepsilon_k \to 0.
\end{aligned}$$

Thus, $f^* = P_{T_1}f^* \quad m-$a.e.

By the uniqueness of T_1-ACIM, all subsequences $\{f_{\varepsilon_{k_i}}\}_{\varepsilon_{k_i}>0}$ of $\{f_\varepsilon\}_{\varepsilon>0}$ have f^* as their common limit point. Hence, $\|f_\varepsilon - f^*\|_1 \to 0$, as $\varepsilon \to 0$. □

7. Appendix

Let

$$\tau = \begin{cases} x(1+2^\alpha x^\alpha) & x\in[0,\frac{1}{2}),\\ g(x) & x \in [\frac{1}{2},1]. \end{cases}$$

We study a deterministic map $\tau : I \to I$, with partition $\mathcal{P} = \{I_1, I_2\}, I_1 = [0,\frac{1}{2}], I_2 = [\frac{1}{2},1], g(\frac{1}{2}) = 0, g'(x) > 1$.

Lemma 7.1. *Let ν be a τ ACIM. Then the support of ν is I.*

Proof. If $g(x)$ is onto, the uniqueness of the $\tau-$ACIM is well known (see [12]). We only consider the case, when $g(1) < 1$. We have $\tau[0,\frac{1}{2}] = [0,1)$. We need to show that for any interval $J \subset I$, there exists an $n \geq 1$ such that $\tau^n(J) \supseteq [0,\frac{1}{2}]$. If $J \supset I_k, k=1,2$, then obviously $\tau(J) \supseteq [0,\frac{1}{2}]$. If $J \subset I_k$. Since $m(\tau(J)) > m(J)$, there exists a $j \geq 1$ such that $\tau^j(J)$ contains $\frac{1}{2}$ in its interior.

Since $\tau^j(J)$ contains the partition point $\frac{1}{2}$ in its interior, i.e. $\tau^j(J) \supset (t_1,t_2)$ with $\frac{1}{2} \in (t_1,t_2)$. Then $\tau[\frac{1}{2},t_2] = [g(\frac{1}{2}), g(t_2)] = [0, g(t_2)]$, which contains the point 0. Then obviously there exists a $l \geq 1$ such that $\tau^{j+l}(J) \supseteq [0,\frac{1}{2}]$.

Let A denote the support of ν. Since A contains an interval J and A is an invariant set $\tau^n(J) \subseteq A, n \geq 1$. Then, $[0,1) \subset A$. Consequently (by invariance) A must contain I. Moreover, $A \subset I$. Therefore, the support of ν is I. □

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